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FAST TRACK COMMUNICATION

Instanton and superconductivity in supersymmetric CP($N-1$) model

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Abstract
The two-dimensional supersymmetric CP($N-1$) model has a striking similarity to the $\mathcal{N}=2$ supersymmetric gauge theory in four dimensions. The BPS mass formula and the curve of the marginal stability (CMS), which exist in the four-dimensional gauge theory, appear in this two-dimensional CP($N-1$) model. These two quantities are derived by a one-dimensional n-vector spin model in the large n limit for the $N=2$ case. This mapping is further investigated at the critical point. An application of the study of the BPS mass formula is proposed to the phenomena of the spin and charge separations in the Higgs phase.

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This communication is the extension of the bosonic CP($N-1$) model, for which the relation to the superconductivity has been studied [1]. The supersymmetric CP($N-1$) model has been applied to the two-dimensional fluctuation phenomena which are related to the Higgs mechanism [2]. There are several examples in condensed matter physics, which have both spin and charge fluctuations, in the strongly correlated systems. For instance, the high-temperature superconductor in the underdoped region may be one example of the systems, where the magnetic spin fluctuations and the gauge field fluctuations become important. Although our analysis is restricted to the phenomenological one, we intend to study universal behaviour of the fluctuations of the bosonic and fermionic excitations based on the supersymmetric model.

The two-dimensional supersymmetric CP($N-1$) model has $\mathcal{N}=2$ supersymmetry and an axial anomaly [3]. A striking similarity between the four-dimensional supersymmetric QCD and the two-dimensional supersymmetric CP($N-1$) model exists [4]. The electric charge and the topological charge give the BPS mass spectra in the four-dimensional supersymmetric QCD [5]. In the four dimension, a weak-coupling region is separated from a strong-coupling region by a curve of marginal stability (CMS), where the bound state becomes marginal and BPS masses can decay. This CMS has also a correspondence in the CP($N-1$) model [4, 6].

In the four-dimensional $\mathcal{N}=2$ gauge theory, the partition function is given by the statistical sum over random partitions, which leads to the prepotential and the spectral curve

[7]. In the two-dimensional supersymmetric CP($N-1$) case, it is desirable to discuss such a partition function from a useful representation. We find a simple model which gives the same effective potential and the curve of marginal stability (CMS) as the two-dimensional supersymmetric CP(1) model. This model is an exactly solvable n-vector spin model, which was studied for the large-order behaviour of the $\frac{1}{n}$ expansion through the instanton analysis [8]. We reinterpret this n-vector model by the analytic continuation of the temperature T, like the Lee–Yang zeroes of the Ising model, in the complex temperature plane.

The supersymmetric CP($N-1$) model is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2g}[(D_\mu Z_i)^\dagger(D_\mu Z_i) - \lambda(Z_i^* Z_i - 1)] + \frac{1}{2g}\left[\mathrm{i}\bar{\psi}^i\gamma_\mu D_\mu\psi_i + \frac{1}{2}(\sigma^2+\pi^2) - \sqrt{\frac{1}{2}}\bar{\psi}^i(\sigma + \mathrm{i}\pi\gamma_5)\psi_i\right], \tag{1}$$

where ψ_i is two-component fermion. The fields λ, σ and π are auxiliary fields. There appears a dynamically generated mass m, which is same for the boson and for the fermion part by the supersymmetry, and it is evaluated by the renormalization group β function in a one-loop order as

$$\Lambda = \mu\,\mathrm{e}^{1-\frac{\pi}{g}}. \tag{2}$$

We replace the renormalization point μ by the twisted mass m, by putting

$$m = \mu. \tag{3}$$

The dynamically generated mass Λ is small when the coupling g is small. In the four-dimensional SQCD, the massive excitations of monopole, dyon and Noether charges are evaluated by the elliptic integrals. The monodoromy of the three singularities at $x = \pm 1$ and $x = u$ determines these mass formula, where u is a moduli parameter, representing the value of the Higgs field. In the two-dimensional CP(1) case, we have a different monodoromy with a single parameter $u = m_1^2 = m_2^2$. The electric charge n_e and topological charge n_t, which are integers, are combined with the mass m and the dual mass m_D to form the charge Z_{n_e,n_t} as

$$Z_{n_e,n_t} = mn_e + m_D n_t. \tag{4}$$

In general, m and m_D are complex numbers, and if m and m_D have same phase, i.e. m/m_D is real number, the bound states of (n_e, n_t) become marginal. This marginal case is represented by a curve in the complex m^2 plane, and called as the curve of marginal stability (CMS), which separates the weak-coupling region and the strong-coupling region. Following the argument of the twisted theory, the vacuum angle θ_{eff} and the coupling constant g_{eff}, which are modified by the quantum corrections from the bare values, are combined as

$$\tau = -\frac{\mathrm{i}}{g_{\mathrm{eff}}} + \frac{\theta_{\mathrm{eff}}}{2\pi}, \tag{5}$$

and the relation to the $\tilde{\Lambda}$, which is modified by the θ_{eff} value, is

$$\frac{\tilde{\Lambda}}{m} = \frac{1}{2}\,\mathrm{e}^{1-\mathrm{i}\pi\tau} = \frac{1}{2}\,\mathrm{e}^{1-\frac{\pi}{g_{\mathrm{eff}}}-\frac{\mathrm{i}}{2}\theta_{\mathrm{eff}}}. \tag{6}$$

There is a critical point at $g_{\mathrm{eff}} = \theta_{\mathrm{eff}} = \pi$, which reads $4\left(\frac{\tilde{\Lambda}}{m}\right)^2 = -1$. The twisted chiral superfield becomes

$$\Sigma = \sigma + \sqrt{2}\vartheta^\alpha\tilde{\chi}_\alpha + \vartheta^\alpha\vartheta_\alpha S. \tag{7}$$

The effective twisted superpotential $\tilde{W}$ with twisted masses m_i [9] is obtained by this twisted chiral superfield Σ. The condition $\partial\tilde{W}/\partial\Sigma = 0$ implies

$$\prod_{i=1}^{N}(\sigma + m_i) - \tilde{\Lambda}^N = \prod_{i=1}^{N}(\sigma - e_i) = 0. \tag{8}$$

For the simplicity, from now on we consider $N = 2$ case. The supersymmetric vacuum state is given by $\sigma = e_1, e_2$ according to the two different boundary conditions at the infinity. Thus we get

$$\begin{aligned} Z_{12} &= 2[\tilde{W}(e_1) - \tilde{W}(e_2)] \\ &= \frac{1}{2\pi}\left[N(e_1 - e_2) - \sum_{i=1}^{2} m_i \ln\left(\frac{e_1 + m_i}{e_2 + m_i}\right)\right]. \end{aligned} \tag{9}$$

By putting $m_1 = -m_2 = -m/2$, we have $\sigma^2 - \frac{m^2}{4} = \tilde{\Lambda}^2$. From (9), m_D is expressed by

$$m_D = \frac{\mathrm{i}}{\pi}\left[\sqrt{m^2 + 4\tilde{\Lambda}^2} + \frac{m}{2}\ln\left(\frac{m - \sqrt{m^2 + 4\tilde{\Lambda}^2}}{m + \sqrt{m^2 + 4\tilde{\Lambda}^2}}\right)\right], \tag{10}$$

since $\sigma = \pm\sqrt{m^2/4 + \tilde{\Lambda}^2}$.

In the strong-coupling region $|m| \ll |\tilde{\Lambda}|$, BPS states becomes only $(n_e = 0, n_t = 1)$ and $(n_e = 1, n_t = -1)$, and $|m_D|$ becomes larger than $|m|$. In the weak-coupling region $|\tilde{\Lambda}| \ll |m|$, these two states are bounded, and other (n_e, n_t) BPS states appear, and m_D can be expanded in the power of $\tilde{\Lambda}/m$, in which the whole instanton contributions appear. There is a boundary, called as the curve of the marginal stability (CMS) in the complex mass parameter m^2, where the restructuring of the BPS states occurs. On this CMS, the masses of dyons and solitons become same as the elementary mass m. The curve of marginal stability is expressed as the following equation [6]:

$$\mathrm{Re}\left[\ln\frac{1 + \sqrt{1 + 4\tilde{\Lambda}^2/m^2}}{1 - \sqrt{1 + 4\tilde{\Lambda}^2/m^2}} - 2\sqrt{1 + 4\tilde{\Lambda}^2/m^2}\right] = 0. \tag{11}$$

For the complex m^2 value at a fixed $\tilde{\Lambda}$, the solution of the above equation gives the curve of marginal stability CMS, which divides the complex m^2 plane into two regions, the weak-coupling and strong-coupling regions. There is a singular point on this CMS, which is the point $4\frac{\tilde{\Lambda}^2}{m^2} = -1$, and the value of m_D in (10) becomes vanishing. This critical point is realized for $g_{\mathrm{eff}} = \theta_{\mathrm{eff}} = \pi$ as shown in (6). When $4\frac{\tilde{\Lambda}^2}{m^2}$ is a positive real number, the solution of the equation for CMS is $4\frac{\tilde{\Lambda}^2}{m^2} = (0.663)^2 = 0.440$.

We now discuss the one-dimensional n-vector model, which has an instanton in the large n limit. This n-vector model has been studied for the large-order behaviour of the $\frac{1}{n}$ expansion in [8]. The large-order behaviour is governed by the instanton. The Hamiltonian $\mathcal{H}$ of this model is

$$\mathcal{H} = -J\sum_{i=1}^{M-1}\vec{S}_i \cdot \vec{S}_{i+1}, \tag{12}$$

with a condition,

$$|\vec{S}_i|^2 = \sum_{m=1}^{n} S_i^2(m) = n. \tag{13}$$

The partition function Z for this model is obtained easily by the integration of the angles between the neighbouring spins

$$Z = \left[\left(\frac{nJ}{2kT} \right)^{1-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) I_{\frac{n}{2}-1}\left(\frac{nJ}{kT}\right) \right]^{M-1}, \tag{14}$$

where $I_\nu(z)$ is a modified Bessel function. We use the parameters $\nu = \frac{n}{2}$ and $Y = \frac{2J}{kT}$ for convenience. This modified Bessel function has an integral representation,

$$I_\nu(\nu Y) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{\nu Y}{2}\right)^\nu 4^\nu \, \mathrm{e}^{-\nu Y} \int_0^\infty \mathrm{e}^{-F(t)} \, \mathrm{d}t, \tag{15}$$

with

$$F(t) = \left(\nu + \tfrac{1}{2}\right) t - \left(\nu - \tfrac{1}{2}\right) \ln(1 - \mathrm{e}^{-t}) - 2\nu Y \, \mathrm{e}^{-t}. \tag{16}$$

In the large ν limit, the saddle point equation becomes $\partial F(t)/\partial t = 0$. The two saddle points t_+ and t_- are

$$\mathrm{e}^{-t_\pm} = \frac{Y - 1 \pm \sqrt{1+Y^2}}{2Y}. \tag{17}$$

Using these values, we find the values of the exponent F,

$$F(t_\pm) = \nu \left(1 - Y \mp \sqrt{1+Y^2} - \ln \frac{-1 \pm \sqrt{1+Y^2}}{2Y^2} \right). \tag{18}$$

The difference becomes

$$F(t_-) - F(t_+) = 2\nu \left(\frac{1}{2} \ln \frac{1 - \sqrt{1+Y^2}}{1 + \sqrt{1+Y^2}} + \sqrt{1+Y^2} \right). \tag{19}$$

Above quantity is known to give the dominant contribution to the large-order behaviour [8]. In the $\frac{1}{n}$ expansion, we obtain

$$I_\nu(\nu Y) = \frac{1}{2\pi\nu} \frac{\mathrm{e}^{\nu\eta}}{(1+Y^2)^{\frac{1}{4}}} \left(1 + \sum_{k=1}^\infty \frac{u_k(t)}{\nu^k} \right), \tag{20}$$

where we obtain η, by collecting the coefficients in (15),

$$\eta = \sqrt{1+Y^2} + \frac{1}{2} \ln \frac{\sqrt{1+Y^2} - 1}{\sqrt{1+Y^2} + 1}. \tag{21}$$

This leading term $\nu\eta$ is different from (19) only by $\nu\pi\mathrm{i}$. The term $u_k(t)$ is determined by the recursion equation [10]. The parameter t is $(1+Y^2)^{-1/2}$. It is an asymptotic expansion, which glows like $(k-1)!/[F(t_+) - F(t_-)]^k$. If we identify $\frac{J}{kT} = \frac{\tilde{\Lambda}}{m}$, $\left(Y^2 = \frac{4\tilde{\Lambda}^2}{m^2}\right)$, we find the expression for m_D/m in (19) except a factor i/π. Expanding the logarithmic term in $F(t_+) - F(t_-)$ by $\sqrt{1+Y^2}$, and taking a double scaling limit of $Y^2 \to -1$ and $N \to \infty$, we obtain the double-scaling relation,

$$F(t_+) - F(t_-) \sim \tfrac{1}{3} n (1+Y^2)^{\frac{3}{2}}. \tag{22}$$

This double-scaling limit corresponds to the superconformal point $\frac{4\tilde{\Lambda}^2}{m^2} = -1$. Note that other models may exist which give the same double-scaling limit, for instance $\lambda\phi^4$ model, but other models do not give the same CMS. The n-vector model is expressed by the Bessel function, which is a confluent hypergeometrical function, and the monodromy of this function is important. This is contrasted with the case of the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, where the periods of the spectral curve are given by the elliptic functions.

Since the correspondence of $J/kT = \tilde{\Lambda}/m$ requires $Y = 2J/kT = \pm\mathrm{i}$ for the superconformal point, we investigate the correlation length of this n-vector model in the large n limit. A one-dimensional n-vector model is solved by the transfer matrix method. The transfer matrix is

$$T = \mathrm{e}^{\frac{nJ}{kT}\vec{S}\cdot\vec{S}'}. \tag{23}$$

The eigenvalues are given by [11]

$$T_l = C I_{\frac{n}{2}-1+l}\left(\frac{nJ}{kT}\right), \tag{24}$$

where $l = 0, 1, 2, \ldots$, and C is a constant. The spin–spin correlation function for the distance r is given by

$$\langle \vec{S}(0)\cdot\vec{S}(r)\rangle = \left(\frac{T_1}{T_0}\right)^r = \left(\frac{I_\nu(\nu Y)}{I_{\nu-1}(\nu Y)}\right)^r, \tag{25}$$

with $\nu = \frac{n}{2}$. If Y is a real, we have a finite correlation length ξ since $I_\nu(\nu Y) < I_{\nu-1}(\nu Y)$, and there is no phase transition at finite temperature. The correlation length ξ is given by

$$\xi^{-1} = \ln\left(\frac{I_{\nu-1}(\nu Y)}{I_\nu(\nu Y)}\right). \tag{26}$$

This quantity becomes positive for the real Y, however, when Y is a pure imaginary number $Y = -\mathrm{i}|Y|$, there appears a phase transition with the infinite correlation length by the degeneracy of the eigenvalues of the transfer matrix. When Y is imaginary, the modified Bessel function is expressed by the Bessel function J as $I_\nu(\nu Y) = \mathrm{e}^{-\nu\pi i/2}J_\nu(\nu|Y|)$. We find in the large ν limit, there appears a crossing at $Y = \pm\mathrm{i}$,

$$J_\nu(\nu|Y|) = J_{\nu-1}(\nu|Y|). \tag{27}$$

The point $Y = \pm\mathrm{i}$ is the critical point, which corresponds to $4\frac{\tilde{\Lambda}^2}{m^2} = -1$. Since the Bessel function $J_\nu(z)$ is an oscillating function, there appear successive degeneracies for ν $\left(\nu = \frac{n}{2} - 1 + l\right)$. The succussive transitions due to the degeneracy of the angular quantum numbers $l = 0, 1, 2, \ldots$, which represent s, p, d, f, ... states. Such successive transitions give a cut in the large n limit beyond $|Y| > 1$, which is a low temperature phase. The mass of the inverse of the correlation length ξ, which is finite in the high temperature region, becomes zero below the transition temperature T_c. It will be interesting to note that such phase transitions also appear for the one-dimensional n-vector model, with a real positive $\frac{J}{kT}$, for $n < 1$, as shown previously in [11]. At the critical point, where the cut appears in the square root singularity in (11), the model may be relevant to the massless Thirring model [12].

We now briefly discuss the relation to the Higgs phase. The CP($N-1$) model has been discussed as an equivalent model of the N-component scalar QED model, or Ginzburg Landau model with a gauge field [1], for the critical behaviour near the transition temperature. The latter is a model for the superconductor, and by the renormalization group analysis in the large N, it has been shown that two model is equivalent when there is a stable fixed point, for instance the critical exponents become same. The superconductor corresponds to $N = 1$ case in the N-component scalar QED model. In this communication, we have discussed a supersymmetric CP($N-1$) model. We have introduced twisted masses, which give the anisotropy for the order parameters. Under this anisotropy, the soliton and dyon (charged soliton) represent the kink singularities, and the instanton becomes a bound state of two opposite kinks. This is similar to the solution of the anisotropic two-dimensional S^2 instanton, which is made of two vortices (merons) [13]. In two dimensions, there is no long-range order, but there is a Kosteritz–Thouless phase. The correlation in the long distance has a behaviour of the algebraic decay, and this corresponds to a vortex bound sate.

The dynamical mass generation of the supersymmetric CP($N-1$) model is interpreted as a formation of a gap. In the four Fermi interaction model, this mass generation is intepreted as a superconductor gap although there is no true long-range order since the dimension is 2. In the supersymmetric CP(1) model with a twisted mass, there is a dynamical generated mass $\tilde{\Lambda}$, which represents a gap. In the weak-coupling case, this gap is made of the bound state of the soliton $(0, 1)$ and dyon $(1, -1)$. It is quite interesting to note that similar bound state is suggested experimentally as a pseudo-gap in the high temperature superconductors. The magnitude of the pseudo-gap is same order as the superconducting gap [14]. If the supersymmetric CP(1) model deformed by a twisted mass is relevant to the high temperature superconductivity phase diagram, the curve of marginal stability CMS, which we have discussed in this communication, may give the boundary curve of the pseudo-gap region.

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